S.-T. Yau College Student Mathematics Contest
Applied and Computational Math (Individual Contest)

June 8, 2024

1. Find an algorithm of $O(n^2)$ operations for solving the following linear system
$$(\boldsymbol{S}\boldsymbol{T} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{b},$$
where $\boldsymbol{S}, \boldsymbol{T} \in \mathbb{R}^{n\times n}$ are given upper triangular matrices, $\lambda \in \mathbb{R}$ is given such that $\boldsymbol{S}\boldsymbol{T} - \lambda\boldsymbol{I}$ is nonsingular, $\boldsymbol{b} \in \mathbb{R}^n$ is given, and $\boldsymbol{x} \in \mathbb{R}^n$ is the unknown vector.

2. Let d^n, g^n and h^n be three non-negative series satisfying
$$\frac{d^{n+1} - d^n}{k} \le g^n d^n + h^n, \ \forall\ n \ge n_0,$$
and
$$\begin{cases} k\sum_{n=k_0}^{N+k_0} g^n \le a_1 \\ k\sum_{n=k_0}^{N+k_0} h^n \le a_2 \quad ,\forall\ k_0 \ge n_0 \\ k\sum_{n=k_0}^{N+k_0} d^n \le a_3 \end{cases}$$
with $kN = r$. Show that
$$d^n \le (a_2 + \frac{a_3}{r}) \exp(a_1), \forall\ n \ge n_0 + N.$$

3. Consider the heat distribution in a rod of length $L > 0$ made of two materials with different heat conductivities, denoted as $a \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$, and distributed alternatively along the rod with a periodicity of $\varepsilon := L/N$, $N \in \mathbb{N}$. The problem can be modelled by the following system
$$\begin{cases} -\dfrac{d}{dx}\left(A_\varepsilon(x)\dfrac{d}{dx}u_\varepsilon(x)\right) = f(x), \quad x \in (0, L) \\ u_\varepsilon(0) = u_\varepsilon(L) = 0, \end{cases} \tag{1}$$

where

$$A_\varepsilon(x) = \begin{cases} a & \text{if } x \in \left(0, \dfrac{\varepsilon}{2}\right) \cup \left(\varepsilon, \dfrac{3}{2}\varepsilon\right) \cup \ldots\ldots \cup \left((N-1)\varepsilon, \dfrac{2N-1}{2}\varepsilon\right), \\ b & \text{otherwise}, \end{cases}$$

and $f(x) \in L^2((0, L))$ signifies a source.

(1) Show that equation (1) has a unique weak solution in $H_0^1((0, L))$.

(2) Show that there is a constant A such that $u_\varepsilon \to u$ weakly in $H_0^1((0, L))$, and u is the solution of the following system:

$$\begin{cases} -\dfrac{d}{dx}\left(A\dfrac{d}{dx}u(x)\right) = f(x), \quad x \in (0, L), \\ u(0) = u(L) = 0. \end{cases} \tag{2}$$

Find the value of A.

4. Let u be the solution to the reaction-diffusion equation

$$u_t = \beta u_{xx} + f(u), \quad \text{in } [0, L] \times (0, T]$$

with the homogeneous Neumann boundary condition. We assume that

A1. The reaction function f satisfies that $f \in C^2(\mathbb{R})$ and $f(0) = 0$.

A2. $\exists K > 0$ such that $|f'(u)| \le K, \quad \forall u \in \mathbb{R}$.

We assume further that the problem is well-posed for a given initial condition $u(x, 0)$.

(a) Consider the following standard forward-in-time and central-in-space discretization

$$v_j^{n+1} = v_j^n + \beta\frac{\Delta t}{(\Delta x)^2}(v_{j-1}^n - 2v_j^n + v_{j+1}^n) + \Delta t f(v_j^n)$$

on a uniform space-time mesh $t_n = n\Delta t$ and $x_j = j\Delta x$. Assume that K is small enough. Derive a sufficient condition on Δt, assuming everything else is fixed, for the numerical stability, in an appropriate sense, of the scheme.

(b) Let $e_j^n := u_j^n - v_j^n$ be the numerical error (where $u_j^n = u(x_j, t_n)$). Show that the scheme is convergent by showing that $\|e^n\| \to 0$, in appropriate norm, as $\Delta t, \Delta x \to 0$.